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# The distribution of symmetric matrix quotients

A.K. Gupta[a,*] and D.G. Kabe[b]

[a] *Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403-0221, USA*

[b] *Department of Math and Computer Science, St. Mary's University, Halifax, NS, Canada*

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**Abstract**

Phillips (J. Multivariate Anal. 16 (1985) 157) generalizes Cramer's (Mathematical Methods of Statistics, Princeton University Press, Princeton, NJ, 1946) inversion formula for the distribution of a quotient of two scalar random variables to the matrix quotient case. However, he gives the result for the asymmetric matrix quotient case. This note extends Phillips' (1985) result to the symmetric matrix quotient case.



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## 1. Introduction

Let $x$ and $y$ be two random variables with the joint characteristic function $\phi(\theta_1, \theta_2)$. Then Cramer [1], and Geary [2] state the following formula for the density $g(z)$ of the quotient $z = x/y$:

$$g(z) = (2\pi i)^{-1} \int_{-\infty}^{\infty} \left[ \frac{\partial \phi(\theta_1, \theta_2)}{\partial \theta_2} \right]_{\theta_2 = -z\theta_1} d\theta_1, \tag{1}$$

where $i = \sqrt{-1}$.

*Corresponding author.

*E-mail address:* gupta@bgnet.bgsu.edu (A.K. Gupta).

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Now Phillips [4] records the following matrix generalization of (1). Given a positive definite symmetric matrix $A : (p+q) \times (p+q)$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11}(p \times p), A_{22}(q \times q), \tag{2}$$

he finds the density of the quotient $X = A_{22}^{-1} A_{21}$. If $\phi(F_{21}, F_{22})$ denotes the joint characteristic function of $A_{21}$ and $A_{22}$, then the density of $X$ is

$$g(X) = (2\pi i)^{-pq} \int_{-\infty}^{\infty} \left\{ \left| \frac{\partial}{\partial F_{22}} \right|^{p} \phi(F_{21}, F_{22}) \right\}_{F_{22} = -\frac{1}{2}(XF'_{21} + F_{21}X')} dF_{21}, \tag{3}$$

where $|\frac{\partial}{\partial F_{22}}| = det(\frac{\partial}{\partial f_{rs}}), f_{rs}$ being the $(r,s)$th element of $F_{22}$, and we assume all integrals to be properly evaluated.

Using (3), and assuming $A$ to have the central Wishart density with $n$ degrees of freedom and population covariance matrix identity for convenience, Phillips [4] illustrates (3) by showing the density of $X$ to be

$$g(X) = K|I + XX'|^{-n}, \tag{4}$$

where $K$ denotes the normalizing constants of density functions in this paper.

However, in multivariate statistical analysis the quotients of two symmetric matrices are often used than the quotients $X$ of the above type. Thus, e.g., if $A$ has the Wishart density

$$g(A) = K \exp\left\{ -\frac{1}{2} tr\, A \right\} |A|^{\frac{1}{2}(n-q-p-1)}, \tag{5}$$

then the canonical correlation matrix $R$ defined by

$$R = A_{11}^{-\frac{1}{2}} A_{12} A_{22}^{-1} A_{21} A_{11}^{-\frac{1}{2}} \tag{6}$$

is a symmetric quotient whose density is desired.

We now proceed to extend (3) to study the density of the random matrix $R$ of type (6) in the next section.

## 2. Symmetric matrix quotient density

We first record the following known formulas. If $Y$ is a $p \times p$ positive definite symmetric matrix and $T$ is another $p \times p$ positive definite symmetric matrix, then we have that

$$\left| \frac{\partial}{\partial Y} \right|^{t} \exp\{-trTY\} = (-1)^{pt} \exp\{-trTY\} |T|^{t}, \tag{7}$$

and hence obviously

$$\left| \frac{\partial}{\partial Y} \right|^{t} |T + Y|^{-\frac{1}{2}n} \propto |T + Y|^{-(\frac{n}{2}+t)}. \tag{8}$$

Now assuming two $p \times p$ random positive definite symmetric matrices $A$ and $B$ to have the joint density $g(A, B)$, we write the density of the matrix $G$ defined by $B = A^{\frac{1}{2}} G A^{\frac{1}{2}}$ to be

$$g(G) = K \int g(A, A^{\frac{1}{2}} G A^{\frac{1}{2}}) |A|^p \, dA, \tag{9}$$

where $|A|^p$ is the Jacobian of transformation from $B$ to $G$ (see [3]).

We now observe that

$$h(A, B) = K[E|A|^p]^{-1} |A|^p g(A, B), \tag{10}$$

where $E(|A|^p)$ denotes the expected value of $|A|^p$, defines a new density whose joint characteristic function $\phi(F_{11}, F_{22})$ is

$$\begin{aligned}
&[E|A|^p]^{-1} \int \exp\{tr(iF_{11}A + iF_{22}B)\} |A|^p g(A, B) \, dA \, dB \\
&\quad = K \int \left| \frac{\partial}{\partial F_{11}} \right|^p \exp\{tr(iF_{11}A + iF_{22}A^{\frac{1}{2}} G A^{\frac{1}{2}})\} g(A, A^{\frac{1}{2}} G A^{\frac{1}{2}}) \, dA \, dG \\
&\quad = K \left| \frac{\partial}{\partial F_{11}} \right|^p \phi(F_{11}, F_{22}),
\end{aligned} \tag{11}$$

where $F_{11}(p \times p)$ and $F_{22}$ are defined in the usual way for evaluation of the characteristic functions of symmetric matrices.

We now assume that the density of $G$ is invariant under the transformation $G \to HGH'$ for any $p \times p$ orthogonal matrix $H$. Thus we may write $HA^{\frac{1}{2}} G A^{\frac{1}{2}} H' = G^{\frac{1}{2}} A G^{\frac{1}{2}}$, in which case (3) modifies to the formula

$$g(G) = K \int \left[ \left| \frac{\partial}{\partial F_{11}} \right|^p \phi(F_{11}, F_{22}) \right]_{F_{11} = -G^{\frac{1}{2}} F_{22} G^{\frac{1}{2}}} dF_{22}. \tag{12}$$

The differentiation in (12) may become involved for the noncentral density of $G$; however, it can be avoided by integrating (11) with respect to $A$ only, and writing (12) as

$$g(G) = K\psi(G) \int [\phi(F_{11}, F_{22}, G)]_{F_{11} = -G^{\frac{1}{2}} F_{22} G^{\frac{1}{2}}} dF_{22}. \tag{13}$$

We illustrate (13) by using (6). The joint characteristic function of $A_{11}$ and $A_{12} A_{22}^{-1} A_{21}$ is

$$\begin{aligned}
\phi(F_{11}, F_{22}) = K \int &\exp\{tr(iF_{11}A + iF_{22}A_{12}A_{22}^{-1}A_{21})\} \exp\left\{ -\frac{1}{2} trA_{11} - \frac{1}{2} trA_{22} \right\} \\
&\times |A_{11} - A_{12}A_{22}^{-1}A_{21}|^{\frac{1}{2}(n-q-p-1)} |A_{22}|^{\frac{1}{2}(n-q-p-1)} \, dA_{11} \, dA_{12} \, dA_{22}.
\end{aligned} \tag{14}$$

Setting $A_{12}A_{22}^{-1}A_{21} = G$, and integrating out $A_{12}, A_{22}$, we reduce (14) to the integral

$$\begin{aligned}\phi(F_{11}, F_{22}, G) =& K \int \exp\{tr(iF_{11}A_{11} + iF_{22}G)\} \exp\left\{-\frac{1}{2}\, trA_{11}\right\} \\ & \times |A_{11} - G|^{\frac{1}{2}(n-q-p-1)}|G|^{\frac{1}{2}(q-p-1)}\, dA_{11}. \end{aligned} \tag{15}$$

Further setting $G = HA_{11}^{\frac{1}{2}}RA_{11}^{\frac{1}{2}}H' = R^{\frac{1}{2}}A_{11}R^{\frac{1}{2}}$, (15) yields the result

$$\begin{aligned}\phi(F_{11}, F_{22}, G) =& K \int \exp\{tr(iF_{11}A_{11} + iR^{\frac{1}{2}}F_{22}R^{\frac{1}{2}}A_{11})\} \exp\left\{-\frac{1}{2}\, trA_{11}\right\} \\ & \times |A_{11}|^{\frac{1}{2}(n-p-1)}\psi(R)\, dA_{11} \\ =& K|I - 2F_{11} - 2R^{\frac{1}{2}}F_{22}R^{\frac{1}{2}}|^{-\frac{1}{2}n}\psi(R), \end{aligned} \tag{16}$$

and hence the density of $R$ is

$$g(R) = K\psi(R) \int |I - 2F_{11} - 2R^{\frac{1}{2}}F_{22}R^{\frac{1}{2}}|^{-\frac{1}{2}n}_{F_{11}=-R^{\frac{1}{2}}F_{22}R^{\frac{1}{2}}}\, dF_{22}, \tag{17}$$

where

$$\psi(R) = K|I - R|^{\frac{1}{2}(n-q-p-1)}|R|^{\frac{1}{2}(q-p-1)}. \tag{18}$$

However, note that

$$|I - 2F_{11} - 2R^{\frac{1}{2}}F_{22}R^{\frac{1}{2}}|^{-n}_{F_{11}=-R^{\frac{1}{2}}F_{22}R^{\frac{1}{2}}} = I, \tag{19}$$

and hence the integration in (17) with respect to $F_{22}$ must be dropped or else assume its value to be some constant. Thus the density of $R$ is given by (18).

The essence of the above example may be stated as follows. If $A$ and $B$ two $p \times p$ positive definite symmetric matrices have the joint density

$$g(A, B) = K \exp\left\{-\frac{1}{2}\, tr(A + B)\right\}|A|^{\frac{1}{2}(n-p-1)}|B|^{\frac{1}{2}(q-p-1)}, \tag{20}$$

then the density of $G = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ is obtained by the evaluation of the integral

$$\begin{aligned}\phi(F_{11}, F_{22}, G) =& K \int \exp\{tr(iF_{11}A + iG^{\frac{1}{2}}F_{22}G^{\frac{1}{2}}A)\} \exp\left\{-\frac{1}{2}\, tr(I + G)A\right\} \\ & \times |A|^{\frac{1}{2}(n+q-p-1)}|G|^{\frac{1}{2}(q-p-1)}\, dA \\ =& K|G|^{\frac{1}{2}(q-p-1)} \int [|I + G - 2F_{11} - 2G^{\frac{1}{2}}|^{-\frac{1}{2}(n+q)}]_{F_{11}=-G^{\frac{1}{2}}F_{22}G^{\frac{1}{2}}}\, dF_{22}, \end{aligned} \tag{21}$$

and dropping the integration with respect to $F_{22}$ if the function under the integral sign is a constant. Thus the density of $G$ is

$$g(G) = K|G|^{\frac{1}{2}(q-p-1)}|I + G|^{-\frac{1}{2}(n+q)}. \tag{22}$$

To evaluate (12) directly, we need multivariate analog of the integrals of the type

$$\int_{-\infty}^{\infty} (q - i\theta)^{-\alpha}(1 + i\theta z)^{-\beta}\, d\theta = [B(\alpha, \beta)]^{-1}(1 + z)^{-(\alpha+\beta)}, \tag{23}$$

which appears to be unknown for the matrix case. However, now from (22) and (23) we conclude that

$$\int |I - iF_{11}|^{-\alpha}|I + iG^{\frac{1}{2}}F_{11}G^{\frac{1}{2}}|^{-\beta}\, dF_{11} = [B_p(\alpha, \beta)]^{-1}|I + G|^{-(\alpha+\beta)}.$$

In fact, Phillips' [4, p. 160, Eq. (12)] integral

$$\int_{-\infty}^{\infty} |A + GG'|^{-n} dG = |A|^{-(n-\frac{1}{2}q)}\pi^{\frac{1}{2}pq}\pi^{\frac{1}{4}p(1-p)}\Gamma_P(n)/\Gamma_P\left(\frac{1}{2}q\right),$$

where $G$ is $p \times q$, $q \geqslant p$, and $A$ is $p \times p$, is a generalized version of the known integral

$$\int_{-\infty}^{\infty} (ax^2 + bx + c)^{-n}\, dx = a^{n-\frac{1}{2}}2^{2n-1}(4ac - b^2)^{\frac{1}{2}-n}B\left(\frac{1}{2}, n - \frac{1}{2}\right).$$

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